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Some consequences of a non-commutative space-time structure

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Abstract. The existence of a fundamental length (or fundamental time) has been conjectured in many contexts. Here we discuss some consequences of a fundamental constant of this type, which emerges as a consequence of deformation-stability considerations leading to a non-commutative space-time structure. This mathematically well defined structure is sufficiently constrained to allow for unambiguous experimental predictions. In particular we discuss the phase-space volume modifications and their relevance for the calculation of the Greisen–Zatsepin–Kuz'min sphere. The (small) corrections to the spectrum of the Coulomb problem are also computed.

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1 Introduction

The idea of modifying the algebra of space-time coordinates, in such a way that they become non-commuting operators, appeared several times in the literature. Probably, the first proposal goes back to Snyder [1] and some field theories in such spaces were studied by Kadishevsky and collaborators [2,3]. Motivated by string theory and quantum gravity or in the mathematical framework of "quantum spaces", these structures have recently been rediscovered and generalized in several ways (see for example [4–7] and references therein).

Associated to the non-commutative space-time effects is also the role played by a fundamental length (or fundamental time) as a new constant of nature. In my opinion, the most satisfactory and model-independent way to approach these problems is through deformation theory and considerations of structural stability of the physical theories.

To study the stability of physical theories one studies the stability (also called rigidity) of its defining Lie algebra. A Lie algebra is said to be stable (or rigid) if any infinitesimal deformation of its structure constants leads to an isomorphic algebra [8–10]. In this setting, the transition from non-relativistic to relativistic and from classical to quantum mechanics may be interpreted as the replacement of two unstable theories by two stable ones; that is, by theories that do not change in a qualitative manner under a small change of parameters. The deformation parameters are $\frac{1}{c}$ (the inverse of the speed of light) and h (the Planck constant). Stability arises from the fact that the algebraic structures are all equivalent for non-zero values of $\frac{1}{c}$ and h. The zero value is an isolated point corresponding to the deformation-unstable classical theories.

A similar stability analysis of relativistic quantum mechanics [11,12] leads to a non-commutative space-time algebra $\Re_{\ell,\infty}$ (on the tangent space)

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= \mathrm{i}(M_{\mu\sigma}\eta_{\nu\rho} + M_{\nu\rho}\eta_{\mu\sigma} \\
&\qquad -M_{\nu\sigma}\eta_{\mu\rho} - M_{\mu\rho}\eta_{\nu\sigma}), \\
[M_{\mu\nu}, p_\lambda] &= \mathrm{i}(p_\mu\eta_{\nu\lambda} - p_\nu\eta_{\mu\lambda}), \\
[M_{\mu\nu}, x_\lambda] &= \mathrm{i}(x_\mu\eta_{\nu\lambda} - x_\nu\eta_{\mu\lambda}), \\
[p_\mu, p_\nu] &= 0, \\
[x_\mu, x_\nu] &= -\mathrm{i}\epsilon\ell^2 M_{\mu\nu}, \\
[p_\mu, x_\nu] &= \mathrm{i}\eta_{\mu\nu}\Im, \\
[p_\mu, \Im] &= 0 \\
[x_\mu, \Im] &= \mathrm{i}\epsilon\ell^2 p_\mu, \\
[M_{\mu\nu}, \Im] &= 0,
\end{aligned} \tag{1}$$

and to two new parameters (ℓ, ϵ), ℓ being a fundamental length (or fundamental time) and ϵ a sign ($\epsilon = -1$ or $\epsilon = +1$). In (1) $\eta_{\mu\nu} = (1, -1, -1, -1)$, $c = \hbar = 1$ and $\Im$ is the operator that replaces the trivial center of the Heisenberg algebra.

The non-commutative space-time geometry arising from this algebra has been studied [13], as well as the modification of the uncertainty relations [14].

Here I will concentrate on some consequences of this non-commutative structure which might lead to simpler experimental tests. In particular phase-space suppression or enhancing effects will be discussed and their relevance to the calculation of the Greisen–Zatsepin–Kuz'min (GZK) sphere [15,16], defined in Sect. 3, as well as the corrections to the spectrum of the Coulomb problem.

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Notice that the modifications introduced on the calculation of the GZK sphere do not arise from violation of Lorentz invariance, which is well preserved, but from a change on the cross sections due to a phase-space volume suppression at high energies. The phase-space suppression only occurs if $\epsilon = +1$. If $\epsilon = -1$ there would be a phase-space enhancing. The $\epsilon = -1$ and $\epsilon = +1$ cases are also quite different as far as the spectrum of the space-time coordinates is concerned. In the first case it is a space coordinate that has a discrete spectrum, whereas the time spectrum is continuous. In the second, it is time that is discrete, space always having a continuous spectrum.

From (1), Lorentz symmetry is preserved in the sense that the commutation relations of the $M_{\mu\nu}$'s are preserved as well as the four-vector nature of both x_μ and p_μ. Notice however that, the coordinates being a non-commutative operator set, they cannot all be diagonalized at the same time. Statements about continuity or discreteness of the spectrum of space or time coordinates only apply when a single component is diagonalized. Therefore it does not make sense to think of Lorentz transformations as mixing the spectrum of different components, because they cannot be simultaneously diagonalized. Lorentz transformations act on the operator space of the x_μ's , generating linear combinations of these operators defining the new coordinate operators in a different frame. Then these new operators will have a continuous or discrete spectrum depending on their time-like or space-like nature and the sign of ϵ.

2 Phase-space effects arising from non-commutativity

Here we see that depending on the sign of ϵ, the available phase-space volume at high momentum contracts or expands. First, this will be shown in the framework of a full representation of the algebra and then, to obtain a simple analytical estimate of the effect, a simpler representation of a subalgebra will be used.

Let

$$\begin{aligned} p_\mu &= \mathrm{i}\frac{\partial}{\partial \xi^\mu}, \\ \Im &= \mathrm{i}\ell\frac{\partial}{\partial \xi^4}, \\ x_\mu &= \mathrm{i}\ell\left(\xi_\mu \frac{\partial}{\partial \xi^4} - \epsilon \xi^4 \frac{\partial}{\partial \xi^\mu}\right), \\ M_{\mu\nu} &= \mathrm{i}\left(\xi_\mu \frac{\partial}{\partial \xi^\nu} - \xi_\nu \frac{\partial}{\partial \xi^\mu}\right) \end{aligned} \tag{2}$$

be a representation of the $\Re_{\ell,\infty}$ algebra (1) by differential operators in a 5-dimensional commutative manifold $M_5 = \{\xi_a\}$ with metric $\eta_{aa} = (1, -1, -1, -1, \epsilon)$

We will now treat the two cases of the sign of ϵ.

Case $\epsilon = -1$

Changing to polar coordinates in the (ξ^1, ξ^4) plane ($\xi^1 = r\cos\theta, \xi^4 = r\sin\theta$) we have

$$\begin{aligned} p^1 &= -\mathrm{i}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right), \\ x^1 &= \mathrm{i}\ell\frac{\partial}{\partial\theta}. \end{aligned} \tag{3}$$

Eigenstates of the x^1 coordinate, with eigenvalue α, are

$$|\alpha\rangle = C_\alpha(r)\exp\left(-\frac{\mathrm{i}}{\ell}\alpha\theta\right); \tag{4}$$

$\theta \in S^1$ and $C_\alpha(r)$ an arbitrary L^2 function of r. Single-valuedness requires $\alpha \in \ell\mathbb{Z}$. That is, each space coordinate has a discrete spectrum.

The eigenstates of p^1 (with eigenvalue k) are

$$|k\rangle = \exp(\mathrm{i}kr\cos\theta). \tag{5}$$

They have a wave function representation in the position basis:

$$\begin{aligned} \langle\alpha|k\rangle &= \int_0^\infty \mathrm{d}r \int_{-\pi}^{\pi} \mathrm{d}\theta C_\alpha^*(r)\, \mathrm{e}^{\mathrm{i}\left(\frac{\alpha}{\ell}\theta + kr\cos\theta\right)} \\ &= 2\pi(\mathrm{i})^{\frac{\alpha}{\ell}} \int_0^\infty \mathrm{d}r C_\alpha^*(r)\, J_{\frac{\alpha}{\ell}}(kr). \end{aligned} \tag{6}$$

To obtain the density of states one imposes periodic boundary conditions in a box of size L, leading to

$$J_0(kr) = (\mathrm{i})^{\frac{L}{\ell}} J_{\frac{L}{\ell}}(kr), \tag{7}$$

and $C_0^*(r) = C_L^*(r)$.

For large k, using the asymptotic expansion for Bessel functions, (7) leads to

$$\begin{aligned} \sqrt{\frac{2}{kr}}\Big\{\cos\left(kr - \frac{\pi}{4}\right) - (\mathrm{i})^{\frac{L}{\ell}}\cos\left(kr - \frac{L}{2\ell}\pi - \frac{\pi}{4}\right) \\ + O\left(|kr|^{-1}\right)\Big\} = 0. \end{aligned} \tag{8}$$

Asymptotically, this is satisfied both for $\frac{L}{\ell} = 2n$, $n \in \mathbb{Z}$ and odd or $\frac{L}{\ell} = 4n$, $n \in \mathbb{Z}$. Therefore, for very large k, no restrictions are put on the k values. It means that the phase volume required for any new k state shrinks as k becomes large. The density of states diverges for large k.

Case $\epsilon = +1$

With hyperbolic coordinates $(\xi^1 = r\sinh\mu, \xi^4 = r\cosh\mu)$ in the (ξ^1, ξ^4) plane,

$$\begin{aligned} p^1 &= \mathrm{i}\left(\sinh\mu\frac{\partial}{\partial r} - \frac{\cosh\mu}{r}\frac{\partial}{\partial\mu}\right), \\ x^1 &= \mathrm{i}\ell\frac{\partial}{\partial\mu}. \end{aligned} \tag{9}$$

The eigenstates of the x^1 coordinate, with eigenvalue α, are

$$|\alpha\rangle = C_\alpha(r)\exp\left(-\frac{\mathrm{i}}{\ell}\alpha\mu\right) \tag{10}$$

Because $\mu \in \mathbb{R}$, in this case the space coordinates have a continuous spectrum. It is the time coordinate that has a discrete spectrum.

The eigenstates of p^1 are

$$|k\rangle = \exp\left(\mathrm{i}kr\sinh\mu\right), \tag{11}$$

with a wave function representation in the position basis

$$\begin{aligned}\langle\alpha|k\rangle &= \int_0^\infty \mathrm{d}r\int_{-\infty}^{\infty}\mathrm{d}\mu C_\alpha^*(r)\,\mathrm{e}^{\mathrm{i}\left(\frac{\alpha}{\ell}\mu+kr\sinh\mu\right)} \\ &= 2\int_0^\infty \mathrm{d}rC_\alpha^*(r)\,K_{\mathrm{i}\frac{\alpha}{\ell}}(kr) \\ &\quad\times\left(\cosh\left(\frac{\alpha\pi}{2\ell}\right)-\sinh\left(\frac{\alpha\pi}{2\ell}\right)\right).\end{aligned} \tag{12}$$

To obtain the density of states one imposes periodic boundary conditions in a box of size L, leading to

$$K_0(kr) = K_{\mathrm{i}\frac{L}{\ell}}(kr)\left(\cosh\left(\frac{L\pi}{2\ell}\right)-\sinh\left(\frac{L\pi}{2\ell}\right)\right) \tag{13}$$

and $C_0^*(r) = C_{\mathrm{L}}^*(r)$.

Using the (large z) asymptotic expansion

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\left(1+\frac{4\nu^2-1}{8z}+O\left(z^{-2}\right)\right) \tag{14}$$

leads for large k to

$$1-\cosh\left(\frac{L\pi}{2\ell}\right)+\sinh\left(\frac{L\pi}{2\ell}\right)+O\left((kr)^{-1}\right)=0. \tag{15}$$

This cannot be satisfied in the $k\to\infty$ limit. It means that the density of states vanishes for large k.

For arbitrary values of k the exact density of states may be obtained from (7) or (13). However, to obtain a simpler, approximate, form for the density of states it is convenient to use the representation of a subalgebra. Namely, for the subalgebra $\left\{x^i, p^i, \Im\right\}$ (i fixed $=1,2$ or 3) one may use

$$\begin{aligned}x^i &= x, \\ p^i &= \frac{1}{\ell}\sin\left(\frac{\ell}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\right), \\ \Im &= \cos\left(\frac{\ell}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\right)\end{aligned} \tag{16}$$

for the $\epsilon=-1$ case and

$$\begin{aligned}x^i &= x, \\ p^i &= \frac{1}{\ell}\sinh\left(\frac{\ell}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\right), \\ \Im &= \cosh\left(\frac{\ell}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}\right)\end{aligned} \tag{17}$$

for the $\epsilon=+1$ case.

Treating each space dimension independently from the others is not guaranteed to lead to an exact result. However, the simple closed form analytical expressions obtained for the density of states are qualitatively identical to the results obtained from (7) and (13) and easier to handle in practical calculations.

The states

$$|p\rangle = \exp\left(\mathrm{i}kx\right) \tag{18}$$

are eigenstates of p^i corresponding to the eigenvalues

$$\begin{aligned}p(k) &= \frac{1}{\ell}\sin(k\ell) \ \text{ for } \epsilon=-1, \\ p(k) &= \frac{1}{\ell}\sinh(k\ell) \ \text{ for } \epsilon=+1.\end{aligned} \tag{19}$$

Posing periodic boundary conditions for $|p\rangle$ on a box of size L implies

$$k = \frac{2\pi}{L}n, \qquad n\in\mathbb{Z}. \tag{20}$$

From $\mathrm{d}p = \frac{\mathrm{d}p}{\mathrm{d}n}\mathrm{d}n$ one obtains for the density of states

$$\begin{aligned}\mathrm{d}n &= \frac{L}{2\pi}\frac{\mathrm{d}p}{\sqrt{1-\ell^2p^2}} \ \text{ for } \epsilon=-1, \\ \mathrm{d}n &= \frac{L}{2\pi}\frac{\mathrm{d}p}{\sqrt{1+\ell^2p^2}} \ \text{ for } \epsilon=+1.\end{aligned} \tag{21}$$

The density of states vanishes when $p\to\infty$ in the $\epsilon=+1$ case and for $\epsilon=-1$ it diverges at $p=\frac{1}{\ell}$ (which is the upper bound of the momentum in this case). This result is consistent with what has been obtained from the asymptotic form of (7) and (13). However, the density of states in (21) is not exact because it is derived from a subalgebra representation, which cannot be lifted in this simple form to a full representation of the algebra .

The modification of the phase-space volume implies corresponding modifications of the cross sections. As an example, to be used in the calculations of the next section, consider the reaction

$$\gamma + p \to \pi + N \tag{22}$$

at high incident proton energy.

Here and in Sect. 3, simple letters are used to denote quantities in the laboratory frame, primed letters for the rest frame of the incident proton and starred letters for the center of mass. Using (21), the modified part of the phase-space integration in the cross section is

$$\begin{aligned}I(\ell) = \int\int &\frac{k_\pi^2\mathrm{d}k_\pi}{\omega_\pi\sqrt{1+\epsilon\ell^2k_\pi^2}}\frac{p_N^2\mathrm{d}p_N}{E_N\sqrt{1+\epsilon\ell^2p_N^2}} \\ &\times\mathrm{d}\Omega_\pi\mathrm{d}\Omega_N\delta^4\left(p_\gamma+p_p-k_\pi-p_N\right).\end{aligned} \tag{23}$$

At high energies, with quantities in the rest frame of the incident proton, one obtains

$$I(\ell) \sim \int_0^{\omega'_\gamma}\frac{k'\left(\omega'_\gamma-k'\right)\mathrm{d}k'}{\sqrt{1+\epsilon\ell^2k'^2}\sqrt{1+\epsilon\ell^2\left(\omega'_\gamma-k'\right)^2}}. \tag{24}$$

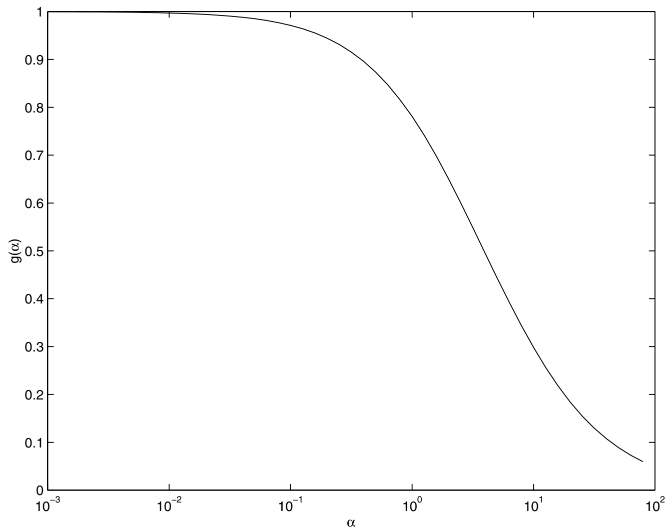

Fig. 1. The phase-space suppression function ($\varepsilon = +1$ case)

Changing variables and dividing by $I(0)$ one obtains the following suppression ($\epsilon = +1$) or enhancing ($\epsilon = -1$) function:

$$g(\alpha,\epsilon) = \frac{I(\ell)}{I(0)} \simeq 6\int_0^1 \frac{x(1-x)\,\mathrm{d}x}{\sqrt{1+\epsilon\alpha x^2}\sqrt{1+\epsilon\alpha(1-x)^2}}, \tag{25}$$

with $\alpha = \omega_\gamma'^2\ell^2$. Figure 1 is a plot of this function in the $\epsilon = +1$ (suppression) case.

3 The GZK sphere

In the sixties, Greisen [15], Zatsepin and Kuz'min [16] have shown that the cosmic microwave background radiation should make the Universe opaque to protons of energies $\gtrsim 10^{20}$ eV. At these energies the thermal photons are sufficiently blue-shifted in the proton rest frame to excite baryon resonances and drain the proton's energy via pion production. This led to the notion of the GZK sphere, the sphere within which a source has to lie to supply us with protons at 10^{20} eV. Later, more accurate calculations, using state-of-the-art particle physics data, placed the energy limit of cosmic (not arising from local sources) protons at around 5.10^{19} eV. That is, if the proton sources are at cosmological distances ($\gtrsim 100$ Mpc), the observed spectrum should display a (GZK) cutoff around this energy. A similar limit applies to the nuclei component of the cosmic ray flux.

This situation was upset by the detection of a number of events above 10^{20} eV without any plausible local sources [17,18] [19]. Discrepancies between the fluxes measured by different groups [20,21] and analysis of the combined data [22] do not yet allow for a clear-cut statement as to whether the GZK cutoff is indeed violated, a question that will hopefully be clarified by the forthcoming Auger observatory. Meanwhile a number of possible explanations for the violation of the GZK cutoff has appeared on the literature (for a review see [23]). Here I analyze the effect of the space-time non-commutativity on the calculation of the GZK cutoff and, when (and if) such cutoff is confirmed, what inferences can be made concerning the value of ℓ and the sign ϵ.

Simple letters are used to denote quantities in the laboratory (earth) frame, primed letters for the rest frame of the proton and starred letters for the center of mass. The fractional energy loss due to interactions with the cosmic background radiation (at zero redshift) is given by the integral of the nucleon energy loss per collision multiplied by the probability per unit time for a nucleon–photon collision in an isotropic gas of photons at temperature $T = 2.7$ K. Therefore the lifetime of a cosmic ray of energy E is [24] ($\hbar = c = 1$)

$$\begin{aligned}\tau_0(E) &= 2\Gamma^2\pi^2 \\ &\times \left\{ \sum_j \int_{\omega'_{j\mathrm{th}}/2\Gamma}^{\infty} \frac{\mathrm{d}\omega}{\mathrm{e}^{\omega/kT}-1} \right. \\ &\qquad \left. \times \int_{\omega'_{j\mathrm{th}}}^{2\Gamma\omega} \mathrm{d}\omega'\,\omega'\,\sigma_j\left(\omega'\right) K_j\left(\omega'\right) \right\}^{-1},\end{aligned} \tag{26}$$

where ω' is the photon energy in the nucleon rest frame and the inelasticity K_j is the average energy lost by the photon for the channel j with threshold $\omega'_{j\mathrm{th}}$. $\sigma_j\left(\omega'\right)$ is the total cross section of the jth interaction channel and Γ the Lorentz factor of the nucleon $\left(\Gamma = \frac{E}{m_p}\right)$.

In (26) one may change the order of integration

$$\int_{\omega'_{\mathrm{th}}/2\Gamma}^{\infty} \mathrm{d}\omega \int_{\omega'_{\mathrm{th}}}^{2\Gamma\omega} \mathrm{d}\omega' \to \int_{\omega'_{\mathrm{th}}}^{\infty} \mathrm{d}\omega' \int_{\omega'/2\Gamma}^{\infty} \mathrm{d}\omega$$

and compute one of the integrals. To obtain the cosmic ray lifetime $\tau_\ell(E)$ in the non-commutative case, one multiplies the cross section by the suppression factor $g(\omega',\epsilon)$ (see (25)). Finally, changing variables

$$\omega' \to y = \mathrm{e}^{-\omega'/(2\Gamma kT)}$$

one obtains the following ratio for each channel contribution:

$$\begin{aligned} r_{\mathrm{g}} &= \frac{\tau_\ell(E)}{\tau_0(E)} \\ &= \int_{\mathrm{e}^{\frac{\omega'_{\mathrm{th}}}{\beta}}}^{0} \frac{\mathrm{d}y}{y} \ln(1-y) \ln y\, \sigma(\beta\ln y)\, K(\beta\ln y) \\ &\quad \Big/ \int_{\mathrm{e}^{\frac{\omega'_{\mathrm{th}}}{\beta}}}^{0} \frac{\mathrm{d}y}{y} \ln(1-y) \ln y\, \sigma(\beta\ln y) \\ &\qquad \times K(\beta\ln y)\, g\left(\beta^2\ell^2\ln^2 y, \epsilon\right), \end{aligned} \tag{27}$$

with $\beta = -2\Gamma kT$.

This ratio was estimated for the single pion reaction (22) using $\omega'_{\mathrm{th}} = 145$ MeV,

$$K(\omega') = \frac{1}{2}\left(1 + \frac{m_\pi^2 - m_N^2}{m_p^2 + 2m_p\omega'}\right),$$

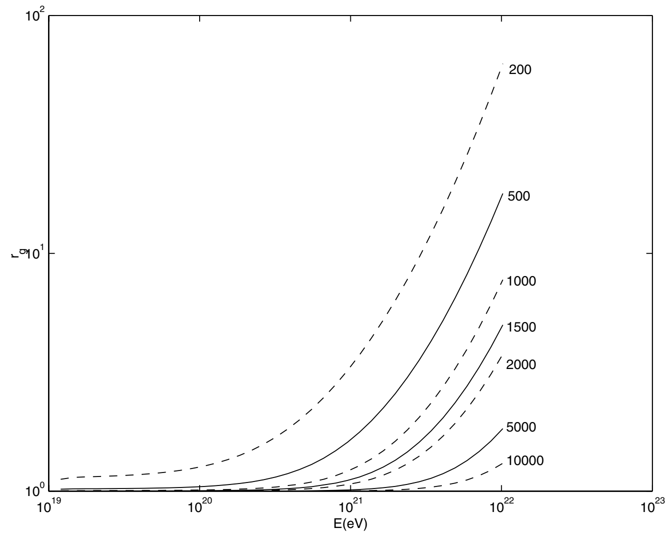

Fig. 2. Cosmic ray lifetime increasing factors for $1/\ell$ in the range 200 to 10 000 MeV ($\varepsilon = +1$ case)

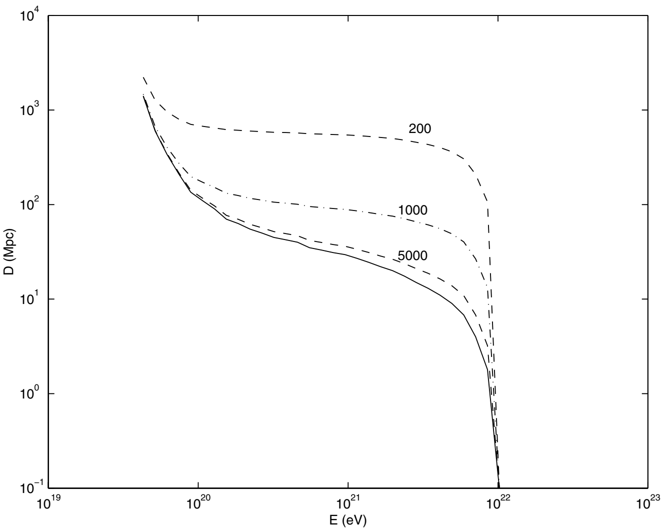

Fig. 3. Energy attenuation of a 10^{22} eV nucleon in route to earth: $1/\ell = 200, 1000, 5000\,\text{MeV}$ compared with the $\ell = 0$ case

and the following parametrization [25] for the cross section:

$$\sigma(\omega') = A + B \ln^2(\omega') + C \ln(\omega'),$$

with $A = 0.147, B = 0.0022, C = -0.017$, and the ω' in GeV. This is a parametrization for the γp total cross section in the range $3\,\text{GeV} < \omega' < 183\,\text{GeV}$. Of course, to compute the absolute value of $\tau_\ell(E)$ this would not be appropriate. Instead, due account should be taken of all the resonance contributions. However for the ratio r_g it gives, at least, qualitative information on the order of magnitude of the effect.

In Fig. 2 the results for r_g are shown for $\epsilon = +1$ and $1/\ell$ in the range 200 to 10 000 MeV, that is, ℓ in the range 0.98–0.0197 Fermi or 329–6.58×10^{-26} s.

To estimate the effect that these lifetime extending factors have on the energy attenuation of cosmic rays on route to earth, I have used the $(\mathrm{d}D/\mathrm{d}E)_\infty$ values found in [26] for a 10^{22} eV nucleon and computed the integration

$$D(E) = D_0(E_0) + \int_{E_0}^{E} r_\text{g}(E)\,(\mathrm{d}D/\mathrm{d}E)_\infty\,\mathrm{d}E. \tag{28}$$

The results are shown in Fig. 3. One sees that whereas the value of the GZK cutoff is not much changed, the radius of the GZK sphere is increased allowing for nucleons from distances beyond 100 Mpc to reach earth at energies above 5.10^{19} eV.

If the observation of the ultra-high energy cosmic rays is indeed a manifestation of the non-commutative structure two conclusions may be drawn.
(1) First, that the sign ϵ is $+1$, that is, space is continuous and time discrete.
(2) Second, that for the effect to be significant at current cosmic ray energies, the time quantum must be $\gtrsim 10^{-25}$ s, much larger than Planck scale times.

It must be pointed out that the idea that modifications to the GZK sphere calculation might arise from the quantum properties of spacetime have already appeared in the literature (see for example [27–31]). They mostly emphasize a possible violation of Lorentz invariance whereas here Lorentz invariance (in the operator sense) is not affected, the modifications coming from changes in the phase-space volume.

4 Corrections to the spectrum of the Coulomb problem

Some experiments in atomic physics are now sensitive to small frequency shifts below 1 mHz. With such sensitivity, non-commutative space-time effects might be detected at low energies, especially if small energy shifts have a qualitative impact. Here such a possibility is analyzed by looking at the effect of the non-commutative algebra on the spectrum of the Coulomb problem. Even if the corrections turn out to be too small to be measurable in the near future, the calculation by itself is of some interest to illustrate how quantum mechanical spectra are computed in the non-commutative framework.

Consider the Hamiltonian

$$H = -\frac{1}{2m}\Delta - \frac{e^2}{|\overrightarrow{x}|}, \tag{29}$$

and use, for the non-commutative coordinates and momenta, the representation listed in the appendix. Both cases ($\epsilon = -1$ and $\epsilon = +1$) will be considered.

$\epsilon = -1$

From (A.3) one obtains (setting $R = 1$)

$$|\overrightarrow{p}|^2 = \frac{1}{\ell^2}\sin^2\theta_3,$$
$$|\overrightarrow{x}|^2 = \ell^2\left\{L^2\cot^2\theta_3 - \frac{\partial^2}{\partial\theta_3^2} - 2\cot\theta_3\frac{\partial}{\partial\theta_3}\right\}. \quad (30)$$

Therefore

$$H = \frac{1}{2m\ell^2}\sin^2\theta_3 - \frac{e^2}{\ell}\left\{L^2\cot^2\theta_3 - \frac{\partial^2}{\partial\theta_3^2} - 2\cot\theta_3\frac{\partial}{\partial\theta_3}\right\}^{-\frac{1}{2}}. \quad (31)$$

For small ℓ, (small θ_3) one obtains

$$H \simeq \frac{1}{2m}\left\{\frac{\theta_3^2}{\ell^2} - \frac{\ell^2}{3}\left(\frac{\theta_3}{\ell}\right)^4\right\} - e^2\left\{L^2\left(\frac{\ell^2}{\theta_3^2} - \frac{2\ell^2}{3}\right) - \ell^2\frac{\partial^2}{\partial\theta_3^2} - 2\ell^2\left(\frac{1}{\theta_3} - \frac{\theta_3}{3}\right)\frac{\partial}{\partial\theta_3}\right\}^{-\frac{1}{2}}. \quad (32)$$

In this approximation $p \simeq \frac{\theta_3}{\ell}$, therefore

$$H \simeq \frac{1}{2m}\left\{p^2 - \frac{\ell^2}{3}p^4\right\} - e^2\left\{L^2\left(\frac{1}{p^2} - \frac{2\ell^2}{3}\right) - \frac{\partial^2}{\partial p^2} - 2\left(\frac{1}{p} - \frac{\ell^2}{3}p\right)\frac{\partial}{\partial p}\right\}^{-\frac{1}{2}},$$

which may be rewritten

$$H \simeq \frac{1}{2m}p^2 - \frac{e^2}{\left(\nabla_p^2\right)^{\frac{1}{2}}} + \ell^2\left\{-\frac{1}{6m}p^4 - \frac{e^2\left(L^2 - p\frac{\partial}{\partial p}\right)}{3\left(\nabla_p^2\right)^{\frac{3}{2}}}\right\}, \quad (33)$$

with $\nabla_p^2 = \frac{L^2}{p^2} - \frac{\partial^2}{\partial p^2} - \frac{2}{p}\frac{\partial}{\partial p}$

Using the Fourier transform $f(x) = \int e^{ip.x}F(p)\,d^3p$ and the relations

$$\int e^{ip.x}p^2F(p)\,d^3p = -\nabla_x^2 f(x),$$
$$\int e^{ip.x}\nabla_p^2F(p)\,d^3p = -x^2 f(x),$$
$$\int e^{ip.x}p\frac{\partial}{\partial p}F(p)\,d^3p = \left(-r\frac{\partial}{\partial r} - 3\right)f(x), \quad (34)$$

one obtains a configuration space representation of (33), namely

$$H \simeq -\frac{\nabla_x^2}{2m} - \frac{e^2}{|x|} - \ell^2\left\{\frac{1}{6m}\nabla_x^4 + \frac{e^2}{3}\frac{L^2 + r\frac{\partial}{\partial r} + 3}{r^3}\right\}. \quad (35)$$

The first two terms are the usual Coulomb Hamiltonian and the third is the order ℓ^2 correction arising from the non-commutative structure.

$$\left\langle n'L'M'\left|H\right|nLM\right\rangle \simeq \delta_{LL'}\delta_{MM'}\left\{E_n\delta_{n',n} + \ell^2\left\langle -\frac{1}{6m}\nabla_x^4 - \frac{e^2}{3}\frac{L(L+1) + r\frac{\partial}{\partial r} + 3}{r^3}\right\rangle_{n',n}\right\}, \quad (36)$$

where $r = |\overrightarrow{x}|$.

$\epsilon = +1$

For the $\epsilon = +1$ case one uses the same representation with the replacements $x^\nu \to ix^\nu, p^\nu \to -ip^\nu, \theta_3 \to i\mu$, to obtain

$$|\overrightarrow{p}|^2 = \frac{1}{\ell^2}\sinh^2\mu, \quad (37)$$
$$|\overrightarrow{x}|^2 = \ell^2\left\{L^2\coth^2\mu - \frac{\partial^2}{\partial\mu^2} - 2\coth\mu\frac{\partial}{\partial\mu}\right\}.$$

Then

$$H = \frac{1}{2m\ell^2}\sinh^2\mu - \frac{e^2}{\ell}\left\{L^2\coth^2\mu - \frac{\partial^2}{\partial\mu^2} - 2\coth\mu\frac{\partial}{\partial\mu}\right\}^{-\frac{1}{2}}, \quad (38)$$

and for small μ

$$H \simeq \frac{1}{2m}\left\{\frac{\mu^2}{\ell^2} + \frac{\ell^2}{3}\left(\frac{\mu}{\ell}\right)^4\right\} - e^2\left\{L^2\left(\frac{\ell^2}{\mu^2} + \frac{2\ell^2}{3}\right) - \ell^2\frac{\partial^2}{\partial\mu^2} - 2\ell^2\left(\frac{1}{\mu} + \frac{\mu}{3}\right)\frac{\partial}{\partial\mu}\right\}^{-\frac{1}{2}},$$
$$H \simeq \frac{1}{2m}\left\{p^2 + \frac{\ell^2}{3}p^4\right\} - e^2\left\{L^2\left(\frac{1}{p^2} + \frac{2\ell^2}{3}\right) - \frac{\partial^2}{\partial p^2} - 2\left(\frac{1}{p} + \frac{\ell^2}{3}p\right)\frac{\partial}{\partial p}\right\}^{-\frac{1}{2}}, \quad (39)$$
$$H \simeq \frac{1}{2m}p^2 - \frac{e^2}{\left(\nabla_p^2\right)^{\frac{1}{2}}} + \ell^2\left\{\frac{1}{6m}p^4 + \frac{e^2\left(L^2 - p\frac{\partial}{\partial p}\right)}{3\left(\nabla_p^2\right)^{\frac{3}{2}}}\right\},$$
$$H \simeq -\frac{\nabla_x^2}{2m} - \frac{e^2}{|x|} + \ell^2\left\{\frac{1}{6m}\nabla_x^4 + \frac{e^2}{3}\frac{L^2 + r\frac{\partial}{\partial r} + 3}{r^3}\right\},$$

the conclusion being that for the $\epsilon = +1$ case the order ℓ^2 correction differs from the $\epsilon = -1$ case by a sign change.

Order ℓ^2 corrections to the spectrum, arising from non-commutativity, have been obtained. Notice that exact results using the Runge–Lenz technique cannot be obtained unless one changes the potential [32].

From (35) and (39) one concludes that the relative corrections to the Coulomb spectrum are of order $\left(\frac{\ell^2}{L^2}\right)$ (L being a typical atomic length scale) with possible enhancements at high angular momentum states. Therefore, for example, for a time quantum of 10^{-24} s ($\frac{1}{\ell} \simeq 658$), the effect is only of order 10^{-10}–10^{-11}. Its experimental detection would indeed be rather difficult but, on the other hand, it is comforting to know that sizable effects in the GZK sphere are not contradicted by present atomic experiments.

5 Conclusions

(1) A non-commutative space-time structure and two constants of nature ℓ and ϵ emerge as natural consequences of deformation theory and stability of the fundamental physical theories. Among other effects, this structure implies a modification of phase-space volume which, in particular, has a bearing on the calculation of the GZK sphere. Lorentz invariance is preserved.
(2) Phase-space suppression effects occur only in the $\epsilon = +1$ case. In this case the time coordinate has a discrete spectrum and space coordinates are continuous.
(3) In addition to changing the cross sections of elementary processes, phase-space counting rules have statistical mechanics consequences which might have had a relevant effect at the first stages of the evolution of the universe.
(4) Phase-space volume modifications, time and space coordinates spectra and modifications of the uncertainty relations are consequences of the non-commutative space-time structure which depend only on its algebraic structure. In this sense they are very robust and provide unambiguous tests of the theory. Other consequences might depend on the particular geometric construction that is built on top of the algebraic structure. For example, for a particular geometrical construction [13] the existence of additional components on gauge fields is an intriguing consequence.

Appendix

For specific calculations it is convenient to use a representation of the space-time algebra ($\varepsilon = -1$ case) in the space of functions on the upper sheet of the cone C^4, with coordinates

$$\begin{aligned}
\xi_1 &= R\sin\theta_3\sin\theta_2\sin\theta_1,\\
\xi_2 &= R\sin\theta_3\sin\theta_2\cos\theta_1,\\
\xi_3 &= R\sin\theta_3\cos\theta_2,\\
\xi_4 &= R\cos\theta_3,\\
\xi_5 &= R,
\end{aligned} \tag{A.1}$$

the invariant measure for which the functions are square-integrable being

$$\mathrm{d}\nu(R,\theta_i) = R^2\sin^2\theta_3\sin\theta_2\mathrm{d}R\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\theta_3. \tag{A.2}$$

On these functions the operators of $\Re_{\ell,\infty}$ act as follows:

$$\begin{aligned}
\ell p^0 &= R,\\
\Im &= R\cos\theta_3,\\
\ell p^1 &= R\sin\theta_3\cos\theta_2,\\
\ell p^2 &= R\sin\theta_3\sin\theta_2\cos\theta_1,\\
\ell p^3 &= R\sin\theta_3\sin\theta_2\sin\theta_1,\\
M^{23} &= -\mathrm{i}\frac{\partial}{\partial\theta_1},\\
M^{12} &= -\mathrm{i}\left(\cos\theta_1\frac{\partial}{\partial\theta_2} - \sin\theta_1\cot\theta_2\frac{\partial}{\partial\theta_1}\right),\\
M^{31} &= \mathrm{i}\left(\sin\theta_1\frac{\partial}{\partial\theta_2} + \cos\theta_1\cot\theta_2\frac{\partial}{\partial\theta_1}\right),\\
\frac{x^0}{\ell} &= -\mathrm{i}\left(-\sin\theta_3\frac{\partial}{\partial\theta_3} + R\cos\theta_3\frac{\partial}{\partial R}\right),\\
\frac{x^1}{\ell} &= \mathrm{i}\left(\cos\theta_2\frac{\partial}{\partial\theta_3} - \sin\theta_2\cot\theta_3\frac{\partial}{\partial\theta_2}\right),\\
\frac{x^2}{\ell} &= \mathrm{i}\left(\cos\theta_1\sin\theta_2\frac{\partial}{\partial\theta_3} + \cos\theta_1\cos\theta_2\cot\theta_3\frac{\partial}{\partial\theta_2}\right.\\
&\qquad\left. - \frac{\sin\theta_1}{\sin\theta_2}\cot\theta_3\frac{\partial}{\partial\theta_1}\right),\\
\frac{x^3}{\ell} &= \mathrm{i}\left(\sin\theta_1\sin\theta_2\frac{\partial}{\partial\theta_3} + \sin\theta_1\cos\theta_2\cot\theta_3\frac{\partial}{\partial\theta_2}\right.\\
&\qquad\left. + \frac{\cos\theta_1}{\sin\theta_2}\cot\theta_3\frac{\partial}{\partial\theta_1}\right),\\
M^{01} &= \mathrm{i}\left(\frac{\sin\theta_2}{\sin\theta_3}\frac{\partial}{\partial\theta_2} - \cos\theta_2\cos\theta_3\frac{\partial}{\partial\theta_3}\right.\\
&\qquad\left. - R\cos\theta_2\sin\theta_3\frac{\partial}{\partial R}\right),\\
M^{02} &= -\mathrm{i}\left(\frac{\cos\theta_1\cos\theta_2}{\sin\theta_3}\frac{\partial}{\partial\theta_2} - \frac{\sin\theta_1}{\sin\theta_2\sin\theta_3}\frac{\partial}{\partial\theta_1}\right.\\
&\qquad + \cos\theta_1\sin\theta_2\cos\theta_3\frac{\partial}{\partial\theta_3}\\
&\qquad\left. + R\cos\theta_1\sin\theta_2\sin\theta_3\frac{\partial}{\partial R}\right),\\
M^{03} &= -\mathrm{i}\left(\frac{\sin\theta_1\cos\theta_2}{\sin\theta_3}\frac{\partial}{\partial\theta_2} + \frac{\cos\theta_1}{\sin\theta_2\sin\theta_3}\frac{\partial}{\partial\theta_1}\right.\\
&\qquad + \sin\theta_1\sin\theta_2\cos\theta_3\frac{\partial}{\partial\theta_3}\\
&\qquad\left. + R\sin\theta_1\sin\theta_2\sin\theta_3\frac{\partial}{\partial R}\right).
\end{aligned} \tag{A.3}$$

For the $\varepsilon = +1$ case, one may work out a similar representation on the $C^{3,1}$ cone with coordinates

$$\begin{aligned}
\zeta_1 &= R\cosh\beta\cos\psi_0,\\
\zeta_2 &= R\cosh\beta\sin\psi_0,\\
\zeta_3 &= R\sinh\beta\sin\psi_1,\\
\zeta_4 &= R\sinh\beta\cos\psi_1,\\
\zeta_5 &= R.
\end{aligned} \tag{A.4}$$

Alternatively we may use the above representation multiplying x^μ by i, p^μ by $-$i and replacing θ_3 by iμ. It is easily seen from (1) that the correct commutation relations, for the $\varepsilon = +1$ case, are obtained.

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